14.3 CONSERVATION OF ENERGY

All energy methods are based on a balance of energy, often referred to as the conservation of energy. In this chapter, only mechanical energy will be considered for this energy balance; that is, the energy developed by heat, chemical reactions, and electromagnetic effects will be neglected. As a result, if a loading is applied *slowly* to a body, then these loads will do ***external work*** U_e as they are displaced. This external work is then transformed into ***internal work*** or strain energy U_i, which is stored in the body. When the loads are removed, the strain energy restores the body to its original undeformed position, provided the material's elastic limit is not exceeded. This conservation of energy for the body can be stated mathematically as

$$U_e = U_i \tag{14–23}$$

Truss. To demonstrate how the conservation of energy applies, we will consider the truss in Fig. 14–18, which is subjected to the load **P**, causing the joint to be displaced Δ. Provided **P** is applied gradually, the external work done by **P** is determined from Eq. 14–2, that is, $U_e = \frac{1}{2}P\Delta$. Assuming that **P** develops an axial force **N** in a particular member, the strain energy stored in this member is determined from Eq. 14–16, that is, $U_i = N^2L/2AE$. Summing the strain energies for all the members of the truss, Eq. 14–23 then requires

$$\frac{1}{2}P\Delta = \sum \frac{N^2L}{2AE} \tag{14–24}$$

Once the internal forces (N) in all the members of the truss are determined and the terms on the right calculated, it is then possible to determine the displacement Δ at the joint where **P** is applied.

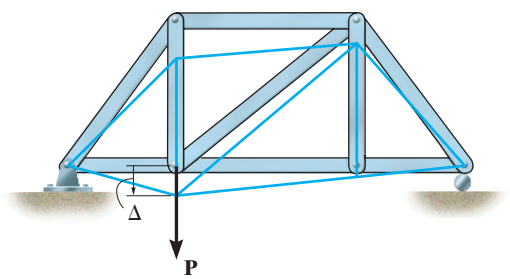

Fig. 14–18

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Beam. Let us now consider finding the vertical displacement Δ under the load **P** acting on the beam in Fig. 14–19. Again, the external work is $U_e = \frac{1}{2}P\Delta$. In this case the strain energy is the result of internal shear and moment loadings caused by **P**. In particular, the contribution of strain energy due to shear is generally *neglected* in most beam deflection problems, unless the beam is short and supports a very large load. (See Example 14.4.) Consequently, the beam's strain energy will be determined only by the internal bending moment M, and therefore, using Eq. 14–17, the conservation of energy requires

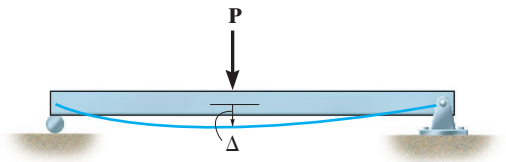

Fig. 14–19

$$\frac{1}{2}P\Delta = \int_0^L \frac{M^2}{2EI}dx \tag{14–25}$$

Once M is expressed as a function of position x and the integral is evaluated, Δ can then be determined.

If the beam is subjected to a couple moment $\mathbf{M}_0$ as shown in Fig. 14–20, then this moment will cause the rotational displacement θ at its point of application. Since a couple moment only does work when it *rotates*, using Eq. 14–5, the external work is $U_e = \frac{1}{2}M_0\,\theta$, and so

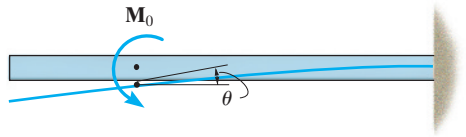

Fig. 14–20

$$\frac{1}{2}M_0\,\theta = \int_0^L \frac{M^2}{2EI}dx \tag{14–26}$$

Here θ measures the *slope* of the elastic curve at the point where $\mathbf{M}_0$ is applied.

Application of Eq. 14–23 for finding a deflection or slope is quite limited, because only a *single* external force or couple moment acts on the member or structure, and only the displacement at the point and in the direction of the external force, or the slope in the direction of the couple moment, can be calculated. If more than one external force or couple moment were applied, then the external work of each loading would have to involve its associated unknown displacement. As a result, none of these unknown displacements could then be determined, since only the single equation ($U_e = U_i$) is available for the solution.

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EXAMPLE 14.6

The three-bar truss in Fig. 14–21*a* is subjected to a horizontal force of 20 kN. If the cross-sectional area of each member is 100 mm^2, determine the horizontal displacement at point B. $E = 200$ GPa.

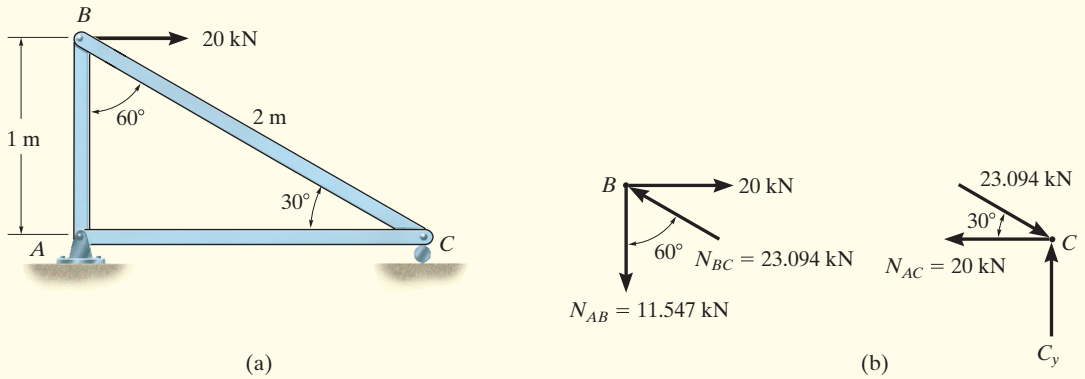

Fig. 14–21

SOLUTION

We can apply the conservation of energy to solve this problem because only a *single* external force acts on the truss and the required displacement happens to be in the *same direction* as the force. Furthermore, the reactive forces on the truss do no work since they are not displaced.

Using the method of joints, the force in each member is determined as shown on the free-body diagrams of the pins at B and C, Fig. 14–21*b*.

Applying Eq. 14–24, we have

$$\frac{1}{2}P\Delta = \sum \frac{N^2 L}{2AE}$$

$$\frac{1}{2}(20\text{ kN})(\Delta_B)_h = \frac{(11.547\text{ kN})^2(1\text{ m})}{2AE} + \frac{(-23.094\text{ kN})^2(2\text{ m})}{2AE} + \frac{(20\text{ kN})^2(1.732\text{ m})}{2AE}$$

$$(\Delta_B)_h = \frac{94.64\text{ kN}\cdot\text{m}}{AE}$$

Notice that since N is squared, it does not matter if a particular member is in tension or compression. Substituting in the numerical data for A and E and solving, we get

$$(\Delta_B)_h = \frac{94.64(10^3)\text{ N}\cdot\text{m}}{[0.1(10^{-3})\text{ m}^2]\,[200(10^9)\text{ N/m}]}$$

$$= 0.004732\text{ m} = 4.73\text{ mm} \rightarrow \qquad \textit{Ans.}$$

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EXAMPLE 14.7

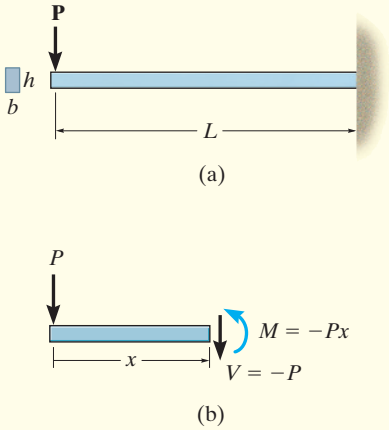

Fig. 14–22

The cantilevered beam in Fig. 14–22*a* has a rectangular cross section and is subjected to a load **P** at its end. Determine the displacement of the load. *EI* is constant.

SOLUTION

The internal shear and moment in the beam as a function of x are determined using the method of sections, Fig. 14–22*b*.

When applying Eq. 14–23 we will consider the strain energy due to both shear and bending. Using Eqs. 14–19 and 14–17, we have

$$\frac{1}{2}P\Delta = \int_0^L \frac{f_s V^2\,dx}{2GA} + \int_0^L \frac{M^2\,dx}{2EI}$$

$$= \int_0^L \frac{\left(\frac{6}{5}\right)(-P)^2 dx}{2GA} + \int_0^L \frac{(-Px)^2 dx}{2EI} = \frac{3P^2L}{5GA} + \frac{P^2L^3}{6EI} \qquad (1)$$

The first term on the right side of this equation represents the strain energy due to shear, while the second is the strain energy due to bending. As stated in Example 14.4, for most beams the shear strain energy is *much smaller* than the bending strain energy. To show when this is the case for the beam in Fig. 14–22*a*, we require

$$\frac{3}{5}\frac{P^2L}{GA} \ll \frac{P^2L^3}{6EI}$$

$$\frac{3}{5}\frac{P^2L}{G(bh)} \ll \frac{P^2L^3}{6E\left[\frac{1}{12}(bh^3)\right]}$$

$$\frac{3}{5G} \ll \frac{2L^2}{Eh^2}$$

Since $E \leq 3G$ (see Example 14.4), then

$$0.9 \ll \left(\frac{L}{h}\right)^2$$

Hence if L is relatively long compared with h, then the shear strain energy can be neglected. In other words, the *shear strain energy* becomes important *only* for *short, deep beams*. For example, if $L = 5h$, then approximately 28 times more bending strain energy will be absorbed in the beam than shear strain energy, so neglecting the shear strain energy represents an error of about 3.6%. With this in mind, Eq. 1 can be simplified to

$$\frac{1}{2}P\Delta = \frac{P^2L^3}{6EI}$$

so that

$$\Delta = \frac{PL^3}{3EI} \qquad \textit{Ans.}$$

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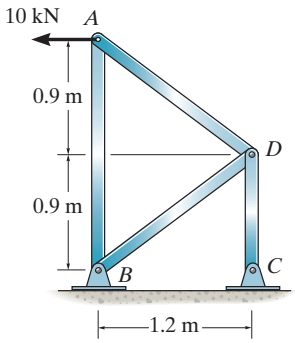
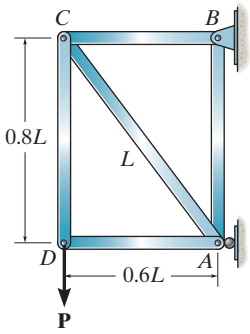
PROBLEMS

14–25. Determine the horizontal displacement of joint A. Each bar is made of A-36 steel and has a cross-sectional area of 950 mm^2.



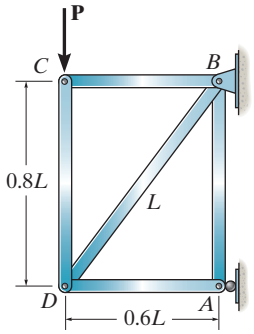
Prob. 14–25

14–26. Determine the vertical displacement of joint D. AE is constant.



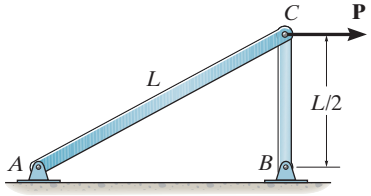
Prob. 14–26

14–27. Determine the vertical displacement of joint C. AE is constant.



Prob. 14–27

*__14–28.__ Determine the horizontal displacement of joint C. AE is constant.



Prob. 14–28

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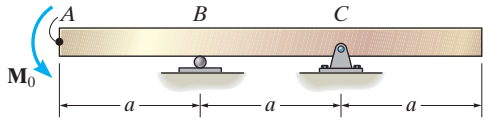
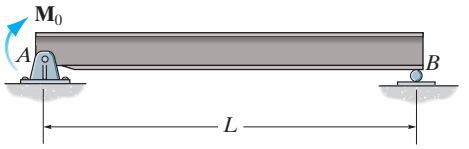
14–29. Determine the slope at point A of the beam. EI is constant.



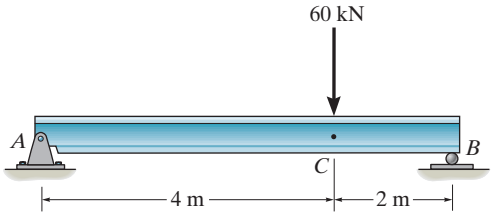
Prob. 14–29

14–30. Determine the slope of the beam at the pin support A. Consider only bending strain energy. EI is constant.



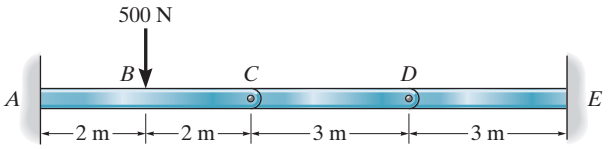
Prob. 14–30

14–31. Determine the vertical displacement of point C of the A992 steel beam. $I = 80(10^6)$ mm^4.



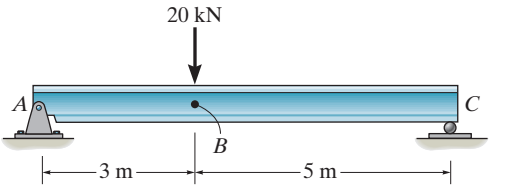
Prob. 14–31

***14–32.** The A992 steel bars are pin connected at C and D. If they each have the same rectangular cross section, with a height of 200 mm and a width of 100 mm, determine the vertical displacement at B. Neglect the axial load in the bars.



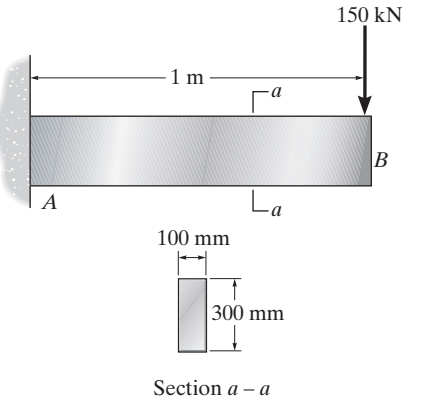
Prob. 14–32

14–33. Determine the vertical displacement of point B on the A992 steel beam. $I = 80(10^6)$ mm^4.



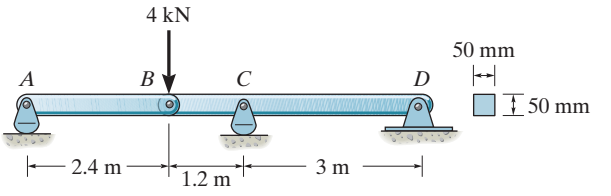
Prob. 14–33

14–34. Determine the vertical displacement of end B of the cantilevered 6061-T6 aluminum alloy rectangular beam. Consider both shearing and bending strain energy.



Prob. 14–34

14–35. The A-36 steel bars are pin connected at B. If each has a square cross section, determine the vertical displacement at B.



Prob. 14–35

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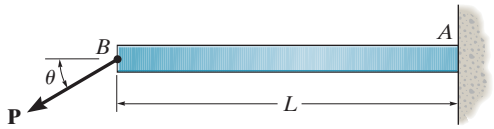
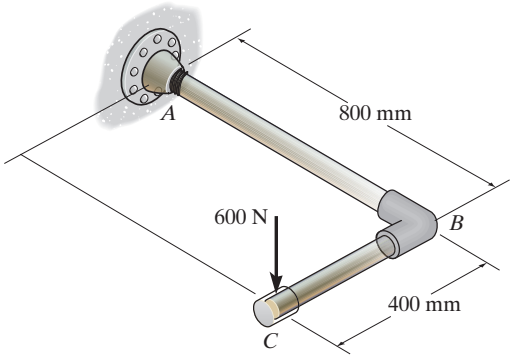
*14–36. The cantilevered beam has a rectangular cross-sectional area A, a moment of inertia I, and a modulus of elasticity E. If a load **P** acts at point B as shown, determine the displacement at B in the direction of **P**, accounting for bending, axial force, and shear.



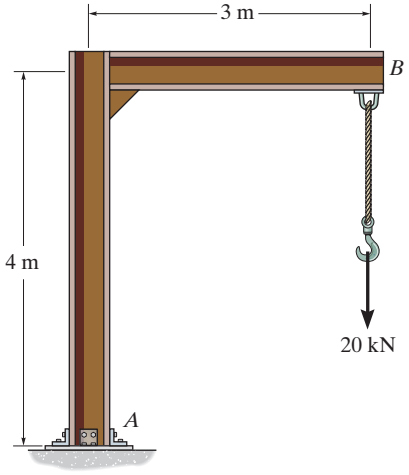
Prob. 14–36

14–37. The pipe assembly is fixed at A. Determine the vertical displacement of end C of the assembly. The pipe has an inner diameter of 40 mm and outer diameter of 60 mm and is made of A-36 steel. Neglect the shearing strain energy.



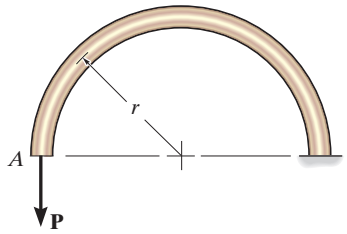
Prob. 14–37

14–38. Determine the vertical displacement of end B of the frame. Consider only bending strain energy. The frame is made using two A-36 steel W460 × 68 wide-flange sections.



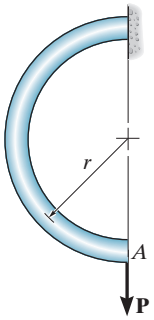
Prob. 14–38

14–39. The rod has a circular cross section with a moment of inertia I. If a vertical force **P** is applied at A, determine the vertical displacement at this point. Only consider the strain energy due to bending. The modulus of elasticity is E.



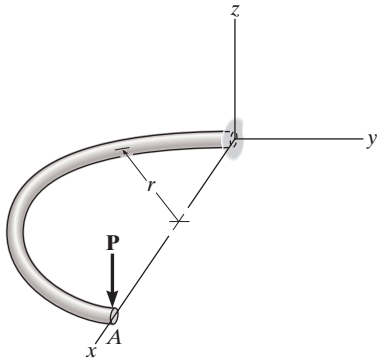
Prob. 14–39

*14–40. The rod has a circular cross section with a moment of inertia I. If a vertical force **P** is applied at A, determine the vertical displacement at this point. Only consider the strain energy due to bending. The modulus of elastcity is E.



Prob. 14–40

14–41. The rod has a circular cross section with a polar moment of inertia J and moment of inertia I. If a vertical force **P** is applied at A, determine the vertical displcement at this point. Consider the strain energy due to bending and torsion. The material constants are E and G.



Prob. 14–41

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